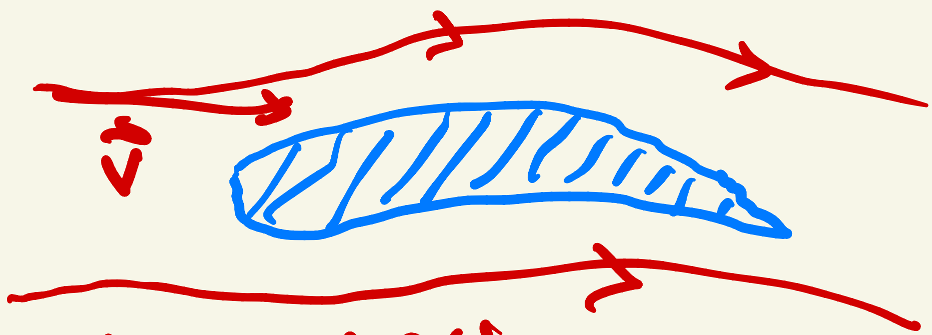
# Math 210 § 15.2

Mass/Density
Center of Mass
Moments of Inertia

start with simpler case of $(x,y) \in \mathbb{R}^2$
density $= \delta(x,y) \in \mathbb{R}$

- Calculus extends the physics of point masses to the continuum
- Basic Set up – Fluid Dynamics

Air Foil



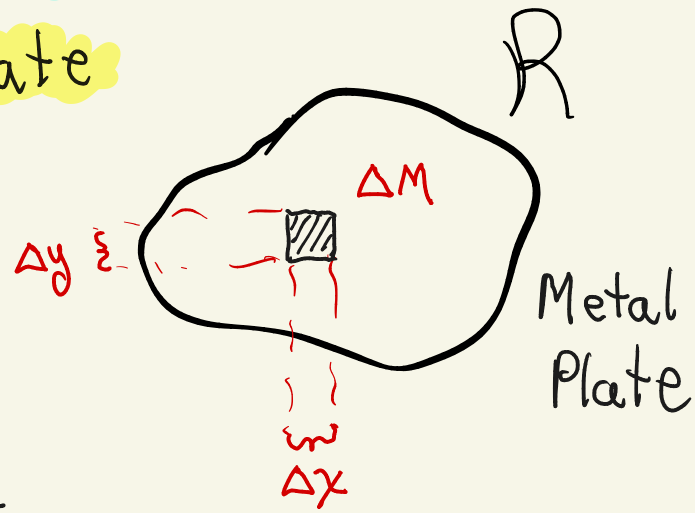

$\delta(x,y) =$ density $= \dfrac{\text{mass}}{\text{area}}$

$\vec{v} = \langle v_1, v_2 \rangle =$ velocity

- Start with the density

$$\delta(x,y) = \frac{\text{Mass}}{\text{Vol}} \equiv \frac{\text{Mass}}{\text{Area}}$$

$\uparrow$ 2-D plate




$$\delta(x,y) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta M}{\Delta x \, \Delta y}$$

Eg: Density can be given by a formula:

$$\delta(x,y) = x y^2 = \frac{\text{mass}}{\text{area}}$$

Q: Given the density, how do you recover the total mass in Region $R$?

- The Idea: Give a precise definition of $\iint_R f(x,y)\,dA$ in terms of a Riemann Sum...

- Identify the total mass as the limit of a Riemann Sum...

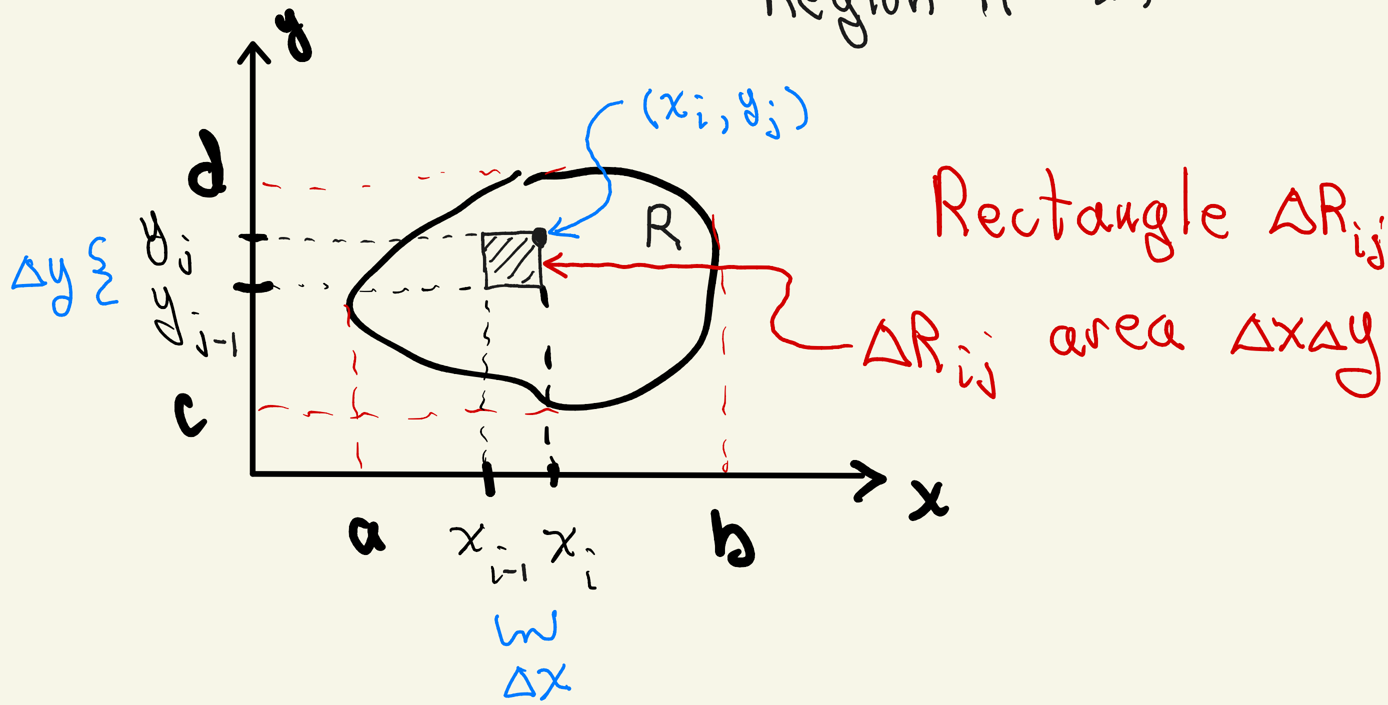
- Evaluate the total mass by evaluating the integral, which is $\iint_R \delta(x,y)\,dA$

- We start by giving a precise definition of the Riemann Integral

$$\iint_R \underbrace{f(x,y)}_{\text{general function } f}\,dA$$

# Defn of $\iint_R f(x,y)\,dA$

Region $R \subseteq [a,b]\times[c,d]$




**Mesh** $a = x_0 < x_1 < \cdots < x_N = b$

$$\Delta x = \frac{b-a}{N}$$

$$c = y_0 < y_1 < \cdots < y_N = d$$

$$\Delta y = \frac{d-c}{N}$$

**Defn**: $$\iint_R f(x,y)\,dA = \lim_{N\to\infty} \underbrace{\sum_{(x_i,y_j)\in R} f(x_i,y_j)\,\Delta x \Delta y}_{\text{Riemann Sum}}$$

$$\iint_R f(x,y)\,dA \equiv \lim_{N\to\infty} \underbrace{\sum_{(x_i,y_j)\in R} \overbrace{f(x_i,y_j)\Delta x\Delta y}^{\text{Approximate volume above } \Delta R_{ij}}}_{\text{Riemann Sum}}$$

Idea: You only sum over rectangles w. mesh points in $R$

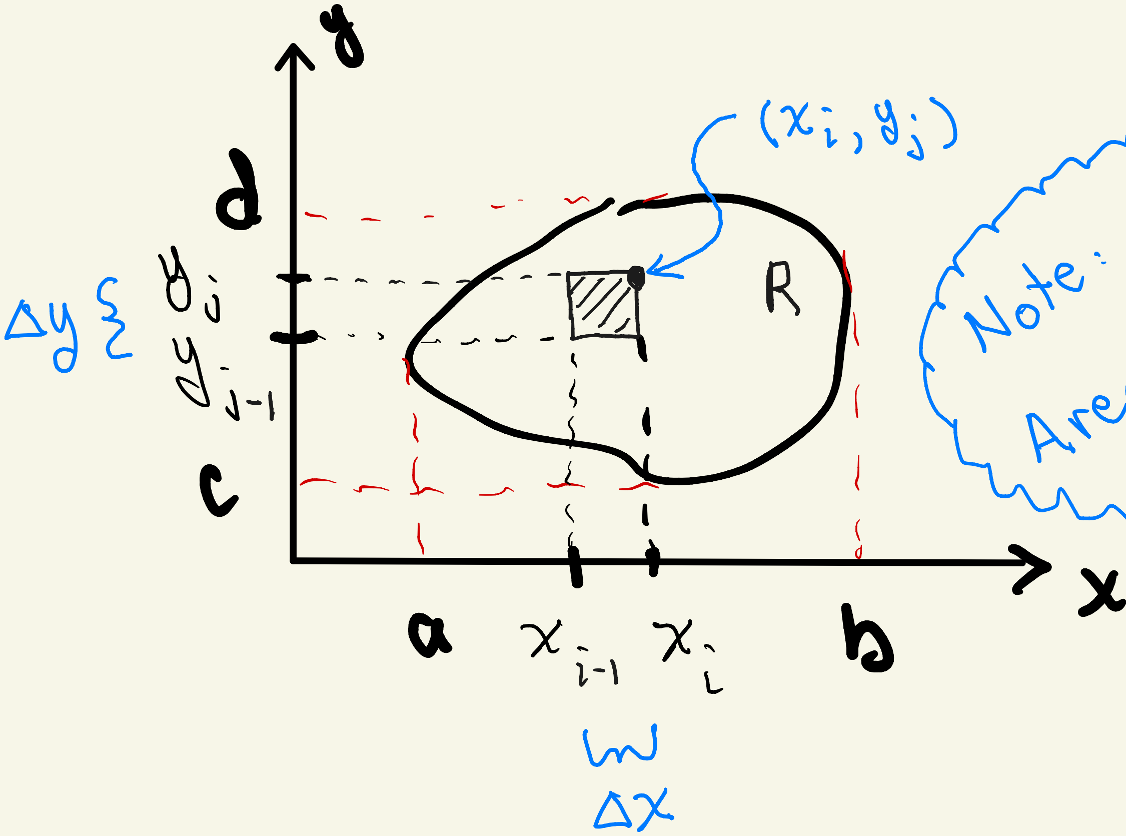
$\rightsquigarrow$ error tends to zero as $N \to \infty$

- Defn of total mass

Ex Let R be a metal plate with density $\delta(x,y) = \frac{\text{mass}}{\text{area}}$

Q: How do you define the Total Mass of R?

Ans: Define it by a Riemann Sum




Note: For Free

$$\text{Area} = \lim_{N\to\infty} \sum_{(x_i,y_j)\in R} \Delta x_i \Delta y_j = \iint_R 1\cdot dA$$

"Mass in Rectangle $\Delta R_{ij}$" $= \Delta M_{ij}$

$$\Delta M_{ij} = \underbrace{\delta(x_i, y_j)}_{\frac{\text{mass}}{\text{area}}} \cdot \underbrace{\Delta x \Delta y}_{\text{Area}}$$

**Defn:** Total Mass $M = \lim_{N\to\infty} \sum_{i,j\in R} \underbrace{\delta(x_i, y_j)\Delta x \Delta y}_{\Delta M_{ij}}$

$$= \iint_R \delta(x,y)\, dA \; !$$

- **Big Idea**: How to go from point masses to the <u>Continuum</u>
  - Identify a continuous physical quantity as the limit of a Riemann Sum
  - Define it to be the integral

Simplest example:

$\delta(x,y)$ = density (continuously distributed mass)

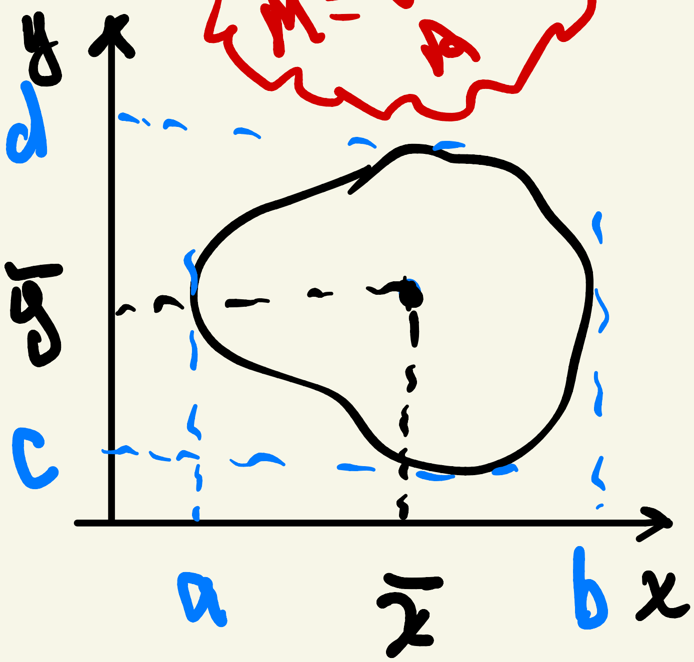
$$\boxed{\text{Total Mass} = \iint_R \delta(x,y)\, dA}$$

▫ **Similarly:** Center of Mass
Moments of Inertia

• Center of Mass

$$M = \iint_A \delta \, dA$$

Q: Given $\delta(x,y)$ what are the coordinates $(\bar{x}, \bar{y})$ of the center of Mass?

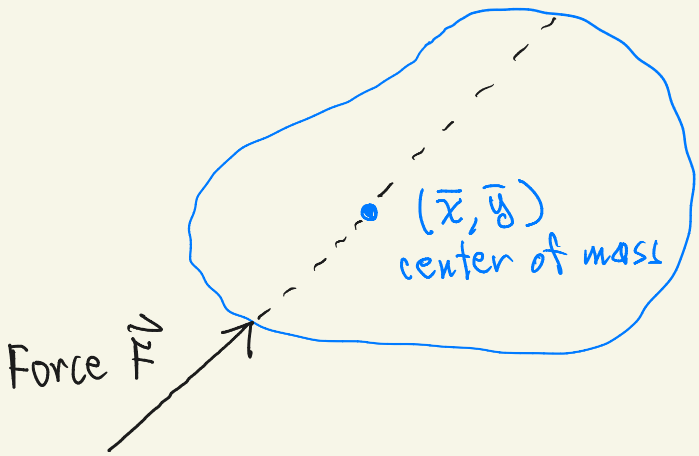



Claim:

$$\bar{x} = \frac{\iint_R x \, \delta(x,y) \, dA}{M}$$

$$\bar{y} = \frac{\iint_R y \, \delta(x,y) \, dA}{M}$$

- **Physically** – The center of mass satisfies the condition that force in line with the center of mass will accelerate the object w/o rotation




$\delta(x,y) = \dfrac{\text{mass}}{\text{Area}}$

Pure acceleration – no rotation

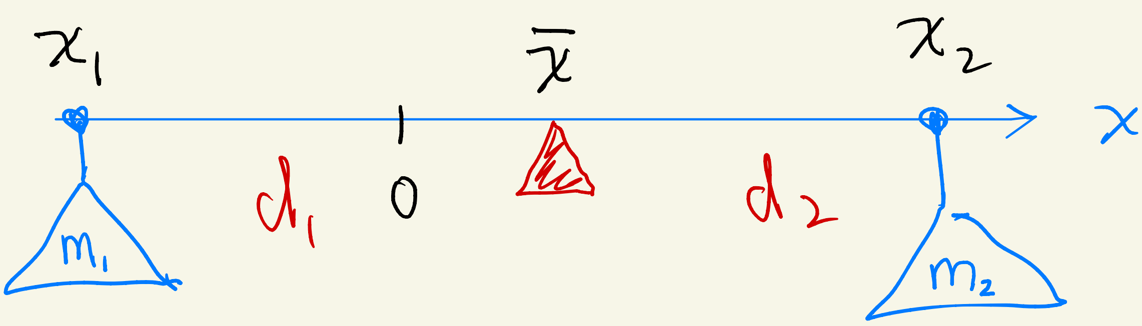
- In this case you will get the same motion assuming all the mass is at $(\bar{x},\bar{y})$

I.e. The point mass $M$ = Total Mass at $(\bar{x},\bar{y})$ moves the same when force is in line with the center of mass –

- A uniform gravitational field "pulls w/o rotating" $\Rightarrow$ moon's orbit is same if you place all mass at CM – Newton invented Calculus to show this!

# ☒ Center of Mass

Ex 1-d Find the point of balance




- At $\bar{x}$: $m_1 d_1 = m_2 d_2$

OR: $(\bar{x} - x_1) m_1 = (x_2 - \bar{x}) m_2$

$$(\bar{x} - x_1) m_1 + \underbrace{(\bar{x} - x_2)}_{\text{neg}} m_2 = 0$$

- Solve for $\bar{x}$ to get CM

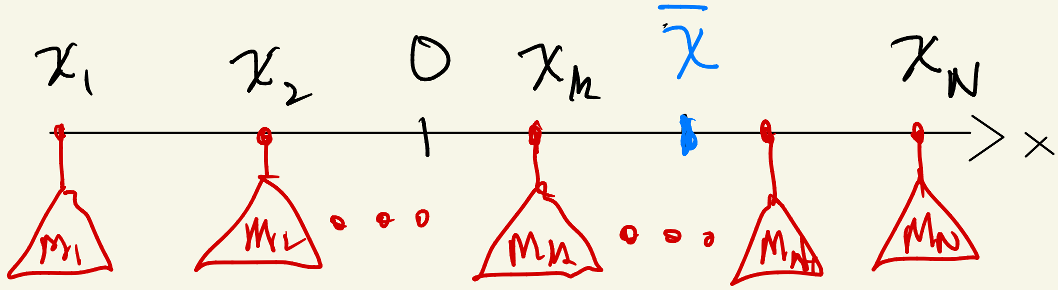
$$\bar{x}(m_1 + m_2) = x_1 m_1 + x_2 m_2$$

$$\bar{x} = \frac{x_1 m_1 + x_2 m_2}{m_1 + m_2}$$

(It didn't matter where I put $x = 0$)

- If there were N-masses

$m_1 \cdots m_N$




# Condition for balance

$$(\bar{x}-x_1)m_1 + (\bar{x}-x_2)m_2 + \cdots + (\bar{x}-x_N)m_N = 0$$

$$\sum_{k=1}^{N} (\bar{x}-x_k)m_k = 0$$

neg to right of $\bar{x}$
pos to left

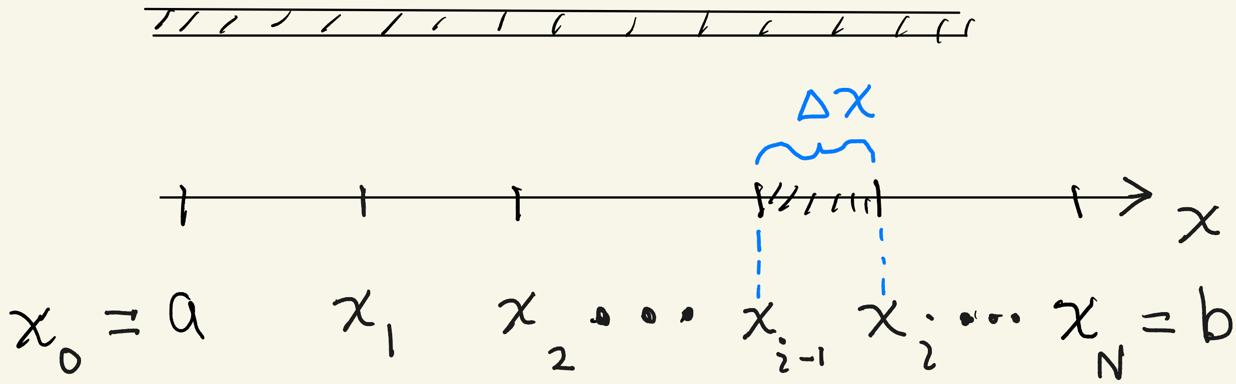
# Solve for $\bar{x}$ !

$$\bar{x}\sum_{k=1}^{N} m_k = \sum_{k=1}^{N} x_k m_k$$

$$\boxed{\bar{x} = \frac{\sum_{k=1}^{N} x_k m_k}{\sum_{k=1}^{N} m_k}}$$

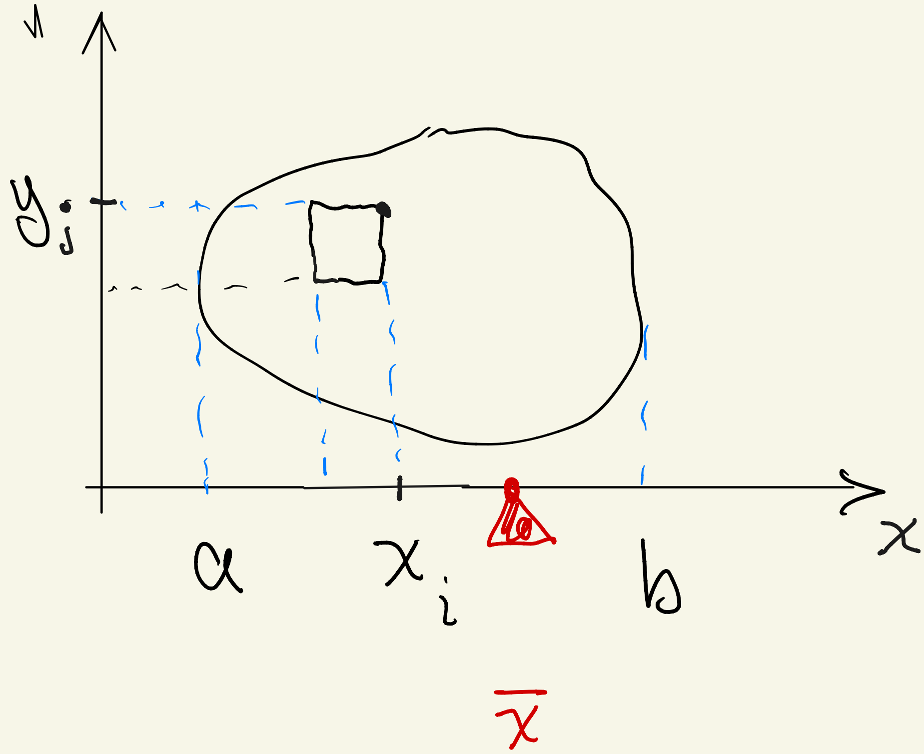
Consider now a density $\delta(x)$

(1-dimension $x \in \mathbb{R}$)




$$\boxed{\begin{aligned} CM &= \lim_{N\to\infty} \frac{\sum_{i=1}^{N} x_i\, \delta(x_i)\,\Delta x}{\sum_{i=1}^{N} \delta(x_i)\,\Delta x} \\ &= \frac{\int_a^b x\, \delta(x)\, dx}{\int_a^b \delta(x)\, dx} = \frac{M_y}{M} \end{aligned}}$$

Similarly: in 2-dimensions $\mathbb{R}^2$ 13




Balance point in $x$ is $\bar{x}$

$$\bar{x} = \lim_{N\to\infty} \frac{\sum x_i \Delta m_i}{\sum \Delta m_i}$$

$$= \lim_{N\to\infty} \frac{\sum_{i,j\in R} x_i \, \delta(x_i, y_j)\Delta x \Delta y}{\sum_{i,j\in R} \delta(x_i, y_j)\Delta x \Delta y}$$

$$= \left(\iint_R x\,\delta(x,y)\,dA\right) \Big/ \left(\iint_R \delta(x,y)\,dA\right) = \frac{M_y}{M}$$

**Conclude:**

$$M_y = \iint_R x\,\delta(x,y)\,dA$$

distance to y-axis is $x$

$$M = \iint_R \delta(x,y)\,dA$$

$$\boxed{\bar{x} = \frac{M_y}{M}, \quad \bar{y} = \frac{M_x}{M}}$$